

Squeezing effect induced by minimal length uncertainty

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In this work, the dynamics of the deformed one-dimensional harmonic oscillator with minimal length uncertainty is examined and the analytical solutions for time evolution of position and momentum operators are presented in which the rough approximation that neglects the higher order terms in BakerHausdor lemma is avoided. Based on these analytical solutions the uncertainties for position and momentum operators are calculated in a coherent state, and an unexpected squeezing effect in both coordinate and momentum directions is found in comparison with ordinary harmonic oscillator. Obviously such a squeezing effect is induced by the minimal length uncertainty (gravitational effects). Our results are applied to the electrons trapped in strong magnetic fields to examine the degree of the existing squeezing effect in a real system, which shows the squeezing degree induced by minimal length uncertainty is very small.

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1. Introduction

The general relativity and quantum mechanics are expected to be unified at Plank scale. The theories that seek to formulate a quantum theory of gravitation are termed as quantum gravity. Almost all the promising approaches to quantum gravity, such as string theory [1], loop quantum gravity [2], and black hole physics [3] suggest the existence of a minimum measurable length. However, such minimal length scale is obviously against the Heisenberg principle $\Delta x \Delta p \geq \hbar/2$, in a way that, arbitrarily precise measurement of position x is no longer possible even at the cost of our knowledge about p . To take into account the effect of minimal length on quantum mechanics, many proposals for quantum gravity suggested a generalized Heisenberg uncertainty principle (GUP) that deforms the canonical one to [4]

$$\Delta \hat{x} \Delta \hat{p} \geq \hbar/2(1 + \beta \Delta \hat{p}^2), \quad (1)$$

where $\beta = \beta_0/(M_{Pl}c)^2 = l_{Pl}^2/2\hbar^2$, β_0 is a parameter that quantifies the correction strength, l_{Pl} is the Plank length. GUP implies the very existence of minimum measurable length predicted by quantum gravity $\Delta x_{\min} = \hbar\sqrt{\beta}$ below which Δx cannot be reduced. Specifically, any quantum state with a position uncertainty Δx inside the region $\Delta x < \Delta x_{\min}$ was proved to have infinite energy, and the expectation value of kinetic energy $\langle E_p \rangle$ in such states is actually divergent regardless of the representation adopted [5]. The emergence of a minimal length scale and GUP will modify all the quantum system with a well-defined Hamiltonian, including low-energy quantum mechanics phenomena. Depending

on the value of the dimensionless parameter β_0 , the unobservable corrections can be interpreted in two different ways. Specifically, one can adopt the regular assumption that β_0 is of the order of unity, in which case the corrections are trivial unless energies (lengths) are near Planck energy (length), or alternatively, set an intermediate length scale between electron weak length scale and Plank scale with a much larger β_0 ($\beta_0 \gg 1$) [4, 6]. Recently, GUP has been extensively studied and numerous proposals have been made to recheck the various quantum systems in the context of GUP [7–11]. It also offers an alternative way to explore quantum gravity effects in terms of measurement of deviation from standard quantum mechanics due to GUP [12, 14].

Harmonic analysis has appeared in a vast range of approaches and techniques in quantum mechanics and quantum optics. Because of its importance as a basic study model, many efforts have been devoted to this subject and the relevant theories are developed to maturity under canonical quantum mechanics frame [13]. Recently, the studies of noncommutative spacetime structures have injected new vitality in this field. Harmonic analysis with minimum length scale can be used as an elementary input for many techniques to address Plank-scale physics using quantum mechanics and quantum optics [14–22]. Considerable physical problems can be regarded as a deformed harmonic oscillator with GUP, such as the oscillations of a carbon monoxide molecule [19], the Landau problem, ultra cold neutrons bouncing above a mirror [10], and singular Calogero potential in one dimension [4, 23]. The problem of harmonic oscillator with the minimal length uncertainty has been considered previously by Kempf et al. [5]. They presented the analytical solution of harmonic oscillator by solving Schrödinger equation, in terms of the energy eigenvalues and eigenstates. Refs. [24, 25] further generalized the analytical result from the one-dimensional to D-dimensional har-

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monic oscillator. Approaches to construct generalized coherent states for harmonic oscillator in the deformed quantum mechanics have been proposed [29, 30]. Ghosh et al. obtained the \hat{x} and \hat{p} uncertainty for the generalized coherent states [31] for a generalized harmonic oscillator, and verified the GUP [32]. However, the results were restricted to a well-designed coherent state with four requirements including temporal stability, which moderated the importance of studies on time evolution. The evolution of position and momentum operators in the context of GUP has been discussed in [20] with classical description. The quantum description has been attempted in [28]. However, the solutions given were based on the approximation that neglects all the higher order terms. The feasibility of the approximation will be broken down at longer time or larger frequency.

In this paper, we first give the analytical expressions for time dependent canonical operators of the deformed harmonic oscillator in section II. In section III, based on the results obtained, we present the analytical solution for time evolution of deformed operators $\hat{x}(t)$, $\hat{p}(t)$ and their uncertainties in a coherent state. Then, we study the temporal behavior of the those uncertainty with the normal quantum mechanics case. Surprisingly, an squeezing effect appears in both position and momentum directions. To further understand this squeezing effect, we use the parameters of electrons trapped in strong magnetic fields to evaluate the significance and magnitude of this effect. In section IV, we end with a conclusion.

2. Time evolution of canonical operators for deformed harmonic oscillations by GUP

The GUP is equivalent to the following modified commutator [26]:

$$[\hat{x}, \hat{p}] = i\hbar(1 + \beta\hat{p}^2). \quad (2)$$

In this paper, we consider the one-dimensional Darboux map [10],

$$\hat{x} = x, \hat{p} = p(1 + \frac{\beta}{3}p^2), \quad (3)$$

where operators with a hat denote the present deformed operators otherwise denote canonical operators. Apparently Eq. (2) is satisfied to $O(\beta)$ with the map (3), thus we neglect terms higher than order β to keep only $O(\beta)$ correction throughout this paper. The Hamiltonian of harmonic oscillator thus becomes

$$H = \frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{x}^2}{2} = \frac{p^2}{2m} + \frac{m\omega^2x^2}{2} + \frac{\beta p^4}{3m}. \quad (4)$$

For our convenience, we introduce canonical annihilation and creation operators

$$a = \sqrt{\frac{m\omega}{2\hbar}}(x + \frac{ip}{m\omega}), a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}(x - \frac{ip}{m\omega}). \quad (5)$$

The Hamiltonian is then rewritten as

$$H = \hbar\omega(a^\dagger a + \frac{1}{2}) + \frac{1}{12}\hbar^2\omega^2 m\beta(a - a^\dagger)^4 \quad (6)$$

In Heisenberg picture of quantum mechanics, motion equation for canonical annihilation operator a is $\frac{da}{dt} = \frac{i}{\hbar}[H, a]$. The time evolution of a is obtained by using Baker-Hausdorff lemma (the lengthy calculation is shown explicitly in the Appendix A).

$$\begin{aligned} a(t) &= e^{\frac{i}{\hbar}Ht} a e^{-\frac{i}{\hbar}Ht} \\ &= a e^{-i\omega t} + \hbar\omega m\beta[-i\omega t e^{-i\omega t}(a + a^\dagger a^2) + \\ &\quad i \sin \omega t(a^\dagger + a^{\dagger 2}a) + \frac{1}{6}(e^{-i\omega t} - e^{-3i\omega t})a^3 \\ &\quad + \frac{1}{12}(e^{-i\omega t} - e^{3i\omega t})a^{\dagger 3}], \end{aligned} \quad (7)$$

and creation operator $a^\dagger(t) = (a(t))^\dagger$. Now let us introduce the following two canonical quadrature operators

$$X_1 = \frac{1}{2}(a(t) + a^\dagger(t)), X_2 = \frac{1}{2i}(a(t) - a^\dagger(t)). \quad (8)$$

Their variances are defined as

$$\begin{aligned} (\Delta X_1)^2 &= \langle X_1^2 \rangle - (\langle X_1 \rangle)^2, \\ (\Delta X_2)^2 &= \langle X_2^2 \rangle - (\langle X_2 \rangle)^2. \end{aligned} \quad (9)$$

We proceed to compute the product of quadrature operators uncertainty $(\Delta X_1)^2(\Delta X_2)^2$ in coherent state $|\alpha\rangle$, which is the eigenstate of canonical annihilation operator a . The computation results are given below (see Appendix B for more details),

$$\begin{aligned} (\Delta X_1)^2 &= \frac{1}{4} + \frac{1}{16}e^{-2i\omega t}\hbar\omega m\beta[(-2 + 3\alpha^2 - 4i\omega t\alpha^2 \\ &\quad + \alpha^{*2} - 8\alpha\text{Re}[\alpha]) + 4e^{2i\omega t}(1 + 2|\alpha|^2) \\ &\quad + e^{4i\omega t}(-2 + \alpha^2 - 4|\alpha|^2 + 4i\omega t\alpha^{*2} - \alpha^{*2})], \end{aligned} \quad (10)$$

$$\begin{aligned} (\Delta X_2)^2 &= \frac{1}{4} + \frac{1}{16}e^{-2i\omega t}\hbar\omega m\beta[(2 - 3\alpha^2 + 4i\omega t\alpha^2 \\ &\quad - \alpha^{*2} + 8\alpha\text{Re}[\alpha]) - 4e^{2i\omega t}(1 + 2|\alpha|^2) \\ &\quad + e^{4i\omega t}(2 - 5\alpha^2 + 4i\omega t\alpha^{*2} - \alpha^{*2})]. \end{aligned} \quad (11)$$

Note that X_1 and X_2 are canonical operators with commutation relation and are supposed to fit the feature of ordinary quantum mechanics regardless of unitary transformation. Thus, the minimum uncertainty requirement of coherent state $(\Delta X_1)^2(\Delta X_2)^2 = \frac{1}{16}$ can be considered as a criterion to justify our computation results to some extent. The product of Eq. (10) and Eq. (11) can exactly meet the requirement.

3. Squeezing effect induced by GUP

With the solution for canonical operators given in Appendix A, it is easy to obtain the time evolution of posi-

tion and momentum in the framework of GUP with the mapping Eq. (3),

$$\begin{aligned}\hat{x}(t) &= x(t) = \sqrt{\frac{2\hbar}{m\omega}}X_1 \\ &= \sqrt{\frac{\hbar}{2m\omega}}(ae^{-i\omega t} + a^\dagger e^{i\omega t}) + \frac{\beta e^{-3i\omega t}}{12} \sqrt{\frac{\hbar^3 m\omega}{2}} [-6e^{2i\omega t}(-1 + e^{2i\omega t} + 2it\omega)a + \\ &\quad 12ie^{3i\omega t}a^\dagger(e^{i\omega t}t\omega + \sin\omega t) + (2e^{2i\omega t} - 3 + e^{4i\omega t})a^3 - (12ie^{2i\omega t}t\omega + 12ie^{3i\omega t}\sin\omega t)a^\dagger a^2 \\ &\quad + (12ie^{4i\omega t}t\omega + 12ie^{3i\omega t}\sin\omega t)a^{\dagger 2}a) + (e^{2i\omega t} + 2e^{4i\omega t} - 3e^{6i\omega t})a^{\dagger 3}],\end{aligned}\quad (12)$$

$$\begin{aligned}\hat{p}(t) &= p(t)(1 + \frac{\beta p^2(t)}{3}) = \sqrt{2\hbar m\omega}(X_2 + \frac{2}{3}\hbar m\omega\beta X_2^3) \\ &= i\sqrt{\frac{\hbar m\omega}{2}}(a^\dagger e^{i\omega t} - ae^{-i\omega t}) - \frac{1}{12\sqrt{2}}e^{-3i\omega t}(\hbar m\omega)^{3/2}\beta[6e^{2i\omega t}(ie^{2i\omega t} + 2t\omega)a \\ &\quad + 6e^{2i\omega t}(-i + 2e^{2i\omega t}t\omega)a^\dagger + (6ie^{4i\omega t} + 12e^{2i\omega t}t\omega)a^\dagger a^2 + (-6ie^{2i\omega t} + 12e^{4i\omega t}t\omega)a^{\dagger 2}a \\ &\quad + i(-3 + 2e^{2i\omega t} - e^{4i\omega t})a^3 + i(e^{2i\omega t} - 2e^{4i\omega t} + 3e^{6i\omega t})a^{\dagger 3}].\end{aligned}\quad (13)$$

Obviously, Eq. (12) and Eq. (13) return to normal case with vanishing β . The dynamics is neither periodic nor harmonic due to the influence of GUP. Note that, the solutions obtained for time evolution of $\hat{x}(t)$ and $\hat{p}(t)$ are different from that given in [28] even expressed in terms of $\hat{x}(0)$ and $\hat{p}(0)$ using Eq. (5). In the framework of first order correction, the results presented in this paper are much more precise. We avoid the controversial approximation adopted in [28] that neglects the terms of order $(\omega t)^5$ and higher. However, our numerical results show that such an approximation is incorrect for a relatively large ωt because in a term with higher order exponential of ωt the increase of the exponential part may overwhelm the decrease of its coefficient, thus the whole term may play a more important role than the term with the lower order exponential of ωt , thus cannot be neglected any more. Next, we proceed to calculate the \hat{x} -variance and \hat{p} -variance in coherent state.

$$(\Delta\hat{x})^2 = (\Delta x)^2 = \frac{2\hbar}{m\omega}(\Delta X_1)^2, \quad (14)$$

$$(\Delta\hat{p})^2 = 2\hbar\omega m(\Delta X_2)^2 + \frac{8}{3}\hbar^2\omega^2 m^2\beta(\langle X_2^4 \rangle - \langle X_2 \rangle \langle X_2^3 \rangle). \quad (15)$$

The derivation for $(\Delta\hat{x})^2$ is straightforward, while to get $(\Delta\hat{p})^2$ we have to calculate $\langle X_2^4 \rangle$ and $\langle X_2^3 \rangle$. The calculation results are complicated and lengthy and here we only directly give the results

$$\begin{aligned}(\Delta\hat{x})^2 &= \frac{\hbar}{2m\omega} + \frac{1}{8}e^{-2i\omega t}\hbar^2\beta[(-2 + 3\alpha^2 - 4i\omega t\alpha^2 \\ &\quad + \alpha^{*2} - 8\alpha\text{Re}[\alpha]) + 4e^{2i\omega t}(1 + 2|\alpha|^2) \\ &\quad + e^{4i\omega t}(-2 + \alpha^2 - 4|\alpha|^2 + 4i\omega t\alpha^{*2} - \alpha^{*2})],\end{aligned}\quad (16)$$

$$\begin{aligned}(\Delta\hat{p})^2 &= \frac{\hbar m\omega}{2} + \frac{1}{8}e^{-2i\omega t}\hbar^2\omega^2 m^2\beta[2 - 7\alpha^2 + 4i\omega t\alpha^2 \\ &\quad - \alpha^{*2} + 8\alpha\text{Re}[\alpha] + e^{4i\omega t}(2 - 5\alpha^2 - 3\alpha^{*2} \\ &\quad - 4i\omega t\alpha^{*2} + 8\alpha\text{Re}[\alpha])].\end{aligned}\quad (17)$$

Then the product of the coordinate and momentum variances follows

$$\begin{aligned}(\Delta\hat{x})^2(\Delta\hat{p})^2 &= \frac{\hbar^2}{4} + \frac{1}{4}\hbar^3 m\omega\beta(1 - e^{-2i\omega t}\alpha^2 + 2|\alpha|^2 - e^{2i\omega t}\alpha^{*2}).\end{aligned}\quad (18)$$

Note that, the generalized Heisenberg uncertainty principle requires $(\Delta\hat{x})^2(\Delta\hat{p})^2 \geq \frac{1}{4}|\langle[\hat{x}, \hat{p}]\rangle|^2$. While

$$\frac{1}{4}|\langle[\hat{x}, \hat{p}]\rangle|^2 = \frac{\hbar^2}{4}(1 + 2\beta\langle\hat{p}^2\rangle), \quad (19)$$

substituting the expression of $\langle\hat{p}^2\rangle$ into Eq. (19) yields exactly the Eq. (18). For the specific expression of $\langle\hat{p}^2\rangle$, we refer the readers to Appendix B. So we conclude

$(\Delta\hat{x})^2(\Delta\hat{p})^2 = \frac{\hbar^2}{4}(1+2\beta\langle\hat{p}^2\rangle)$, that is, the deformed harmonic oscillator with GUP in coherent states still satisfies the minimum uncertainty relation to the first order correction.

To further show how the minimal length affects the dynamics of harmonic oscillator, we compare the results in the framework of GUP with a simple harmonic oscillator in the context of ordinary quantum mechanics with the position and momentum operators denoted by x_0 and p_0 respectively. The variances of canonical operators x_0 and p_0 are $(\Delta x_0)^2 = \frac{\hbar}{2m\omega}$, $(\Delta p_0)^2 = \frac{\hbar m\omega}{2}$, and their product is $(\Delta x_0)^2(\Delta p_0)^2 = \frac{\hbar^2}{4}$. This is actually the Eq.(16)-(18) when $\beta = 0$. Assuming eigenvalue $\alpha = \gamma e^{i\theta}$, where γ is positive and $\theta \in [0, 2\pi]$, we calculate the difference between the variances in GUP context and the corresponding canonical variances.

$$\begin{aligned} & (\Delta\hat{x})^2 - (\Delta x_0)^2 \\ &= \frac{\hbar^2\beta}{2}[2\sin^2\omega t + \gamma^2(4\sin^2\omega t - 2\omega t \cos 2\theta \sin 2\omega t \\ &+ (2\omega t \cos 2\omega t - \sin 2\omega t) \sin 2\theta)], \end{aligned} \quad (20)$$

$$\begin{aligned} & (\Delta\hat{p})^2 - (\Delta p_0)^2 \\ &= \frac{1}{4}\hbar^2 m^2 \omega^2 \beta [2\cos 2\omega t + \gamma^2(4\cos 2\omega t - 3\cos(2\omega t - 2\theta) \\ &- \cos(2\omega t + 2\theta) + 4\omega t \sin(2\omega t - 2\theta))], \end{aligned} \quad (21)$$

$$(\Delta\hat{x})^2(\Delta\hat{p})^2 - (\Delta x_0)^2(\Delta p_0)^2 = 1 + 2\gamma^2(1 - \cos(2\omega t - 2\theta)). \quad (22)$$

Clearly, Eq. (22) indicates the product $(\Delta\hat{x})^2(\Delta\hat{p})^2$ is always larger than $(\Delta x_0)^2(\Delta p_0)^2$ irrespective of the value of parameters, which is demanded by GUP. Nevertheless, Eq. (20) and Eq. (21) cannot guarantee that the deformed variances are invariably larger than their ordinary correspondence. Specifically, by fixing γ , which generally has an overall influence on amplitude, we plot the results of Eq. (20) and (21) as a function of parameters ωt and θ in Fig. 1 and Fig. 2, respectively, where the coefficients containing β , \hbar and m outside bracket in the r.h.s. are incorporated into the vertical coordinate, considering they are positive and do not affect the sign of the outcome. Surprisingly, Fig. 1 and Fig. 2 show that the deformed variances $(\Delta\hat{x})^2$ and $(\Delta\hat{p})^2$ can be smaller than $(\Delta x_0)^2$ and $(\Delta p_0)^2$, respectively. That is to say, squeezing effect unexpectedly emerges in both x and p direction at certain time during the evolution. To study the degree of such squeezing effect, now we apply our results to a real system, an electron in a constant magnetic field. The

cyclotron motion of the electron is actually a harmonic oscillator. By measuring the energy shift of Landau levels with a STM, an upper bound $\beta_0 < 10^{50}$ can be defined [4]. The parameters are chosen as following: cyclotron frequency $\omega_c \approx 10^3$ GHz, $n = 2$, $\alpha = e^{i\frac{\pi}{4}}$, and taking the largest allowed $\beta = \beta_0/(M_{Pl}c)^2 = 2.43478 \times 10^{48}$. With those parameters, we are able to give a more intuitive comparison between the standard and the deformed harmonic oscillator based on uncertainty of position and momentum, as depicted in Fig. 3. The picture shows that, for coherent state, both $(\Delta\hat{x})^2$ and $(\Delta\hat{p})^2$ oscillate, respectively, around the straight lines of $(\Delta x_0)^2 = \frac{\hbar}{2m\omega}$ and $(\Delta p_0)^2 = \frac{\hbar m\omega}{2}$. Even though their product always oscillates above the constant value of $(\Delta x_0)^2(\Delta p_0)^2$. That is, squeezing effect emerges in both x and p direction. But on the other hand, Fig. 3 (a) and (b) show that squeezing degree for both $(\Delta\hat{x})^2$ and $(\Delta\hat{p})^2$ is very small under the chosen parameters. Actually, a much smaller β towards Plank-scale modifications will render the squeezing effects negligible.

4. Conclusion

In conclusion, we have presented the analytical expressions in Heisenberg picture for time evolution of the operators of position \hat{x} and momentum \hat{p} for the deformed harmonic oscillator with GUP, based on which the uncertainties $(\Delta\hat{x})^2$ and $(\Delta\hat{p})^2$ for both position and momentum are calculated in a coherent state. We surprisingly find $(\Delta\hat{x})^2$ and $(\Delta\hat{p})^2$ can be smaller than that of a standard harmonic oscillator in a coherent state, implying the minimal length uncertainty or gravitational effects can induce squeezing effect in both position and momentum. As an example, we apply our results to an electron trapped in strong magnetic fields by taking into account influence of the minimal length uncertainty and find the existing squeezing effect is actually very small. Our results may be useful for in some techniques and approaches to explore and measure the potential quantum gravitational phenomena with such as mechanical oscillator and trapped ions.

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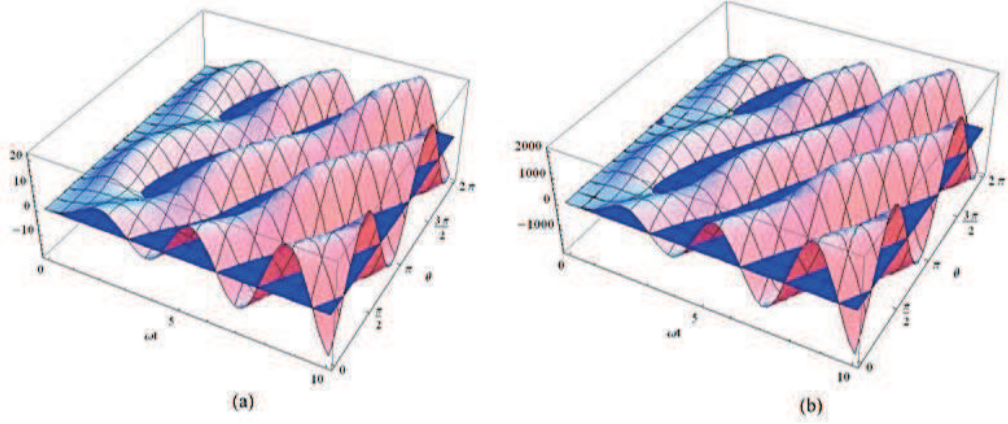


FIG. 1: (color online) The variation of $\frac{2}{\hbar^2 \beta} [(\Delta \hat{x})^2 - (\Delta x_0)^2]$ with ωt and θ . The part that the difference below zero plane is where $(\Delta \hat{x})^2 < (\Delta x_0)^2$, which implicates the appearance of squeezing. $\gamma = 1$ and $\gamma = 10$ in (a) and (b) respectively.

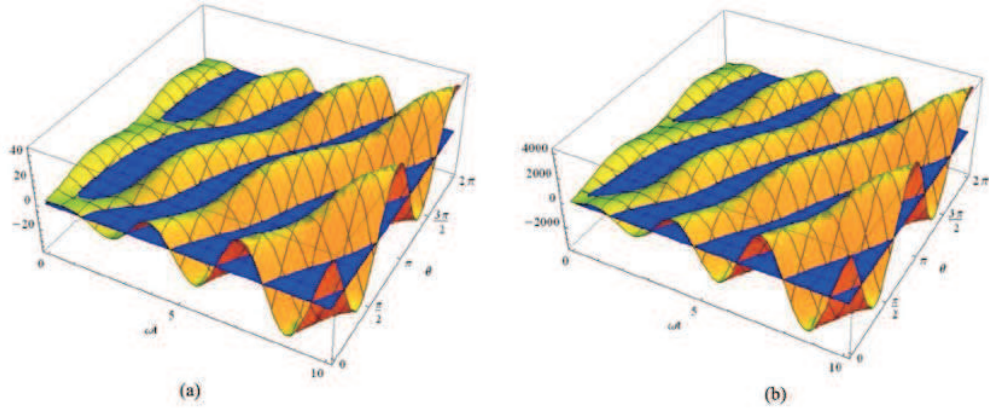


FIG. 2: (color online) The evolution of $\frac{4}{\hbar^2 m^2 \omega^2 \beta} [(\Delta \hat{p})^2 - (\Delta p_0)^2]$ with ωt and θ . The part that the difference below zero plane is where $(\Delta \hat{p})^2 < (\Delta p_0)^2$, which implicates the appearance of squeezing. $\gamma = 1$ and $\gamma = 10$ in (a) and (b) respectively.

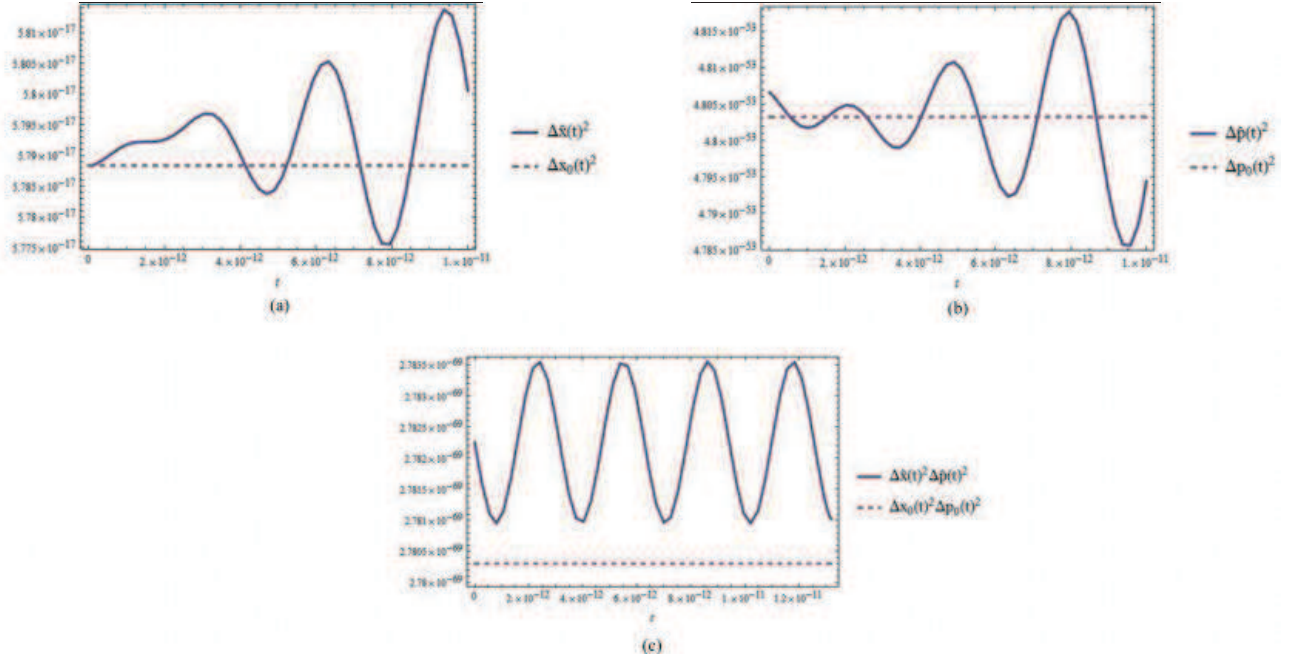


FIG. 3: (color online) The temporal behavior of variances of position (a), momentum (b) their product (c) in a coherent state. The solid and dashed lines respectively stand for deformed and canonical operators.

Appendix A: Derivation for Eq.(7)

The time evolution of a can be obtained by using Baker-Hausdorff lemma

$$e^{\xi A} B e^{-\xi A} = B + \xi[A, B] + \frac{\xi^2}{2!}[A, [A, B]] + \dots \quad (\text{A1})$$

Let $\xi = \frac{i}{\hbar}t$, $A = H$, $B = a$, the second term of the r.h.s. of Eq. (A1) is

$$\xi[A, B] = -it\omega a + i\hbar m\omega^2\beta(-a + a^\dagger + \frac{1}{3}(a^3 - a^{\dagger 3}) - a^\dagger a^2 + a^{\dagger 2}a). \quad (\text{A2})$$

The third term is

$$\frac{\xi^2}{2!}[A, [A, B]] = \frac{(it\omega)^2}{2!}a + \frac{(it)^2}{2!}\hbar m\omega^3\beta(2a - \frac{4}{3}a^3 - \frac{4}{3}a^{\dagger 3} + 2a^\dagger a^2). \quad (\text{A3})$$

The fourth term is

$$\frac{\xi^3}{3!}[A, [A, [A, B]]] = -\frac{(it\omega)^3}{3!}a + \frac{(it)^3}{3!}\hbar m\omega^4\beta(-3a + a^\dagger + \frac{13}{3}a^3 - \frac{7}{3}a^{\dagger 3} - 3a^\dagger a^2 + a^{\dagger 2}a). \quad (\text{A4})$$

The fifth term is

$$\frac{\xi^4}{4!}[A, [A, [A, [A, B]]]] = \frac{(it\omega)^4}{4!}a + \frac{(it)^4}{4!}\hbar m\omega^5\beta(4a - \frac{40}{3}a^3 - \frac{20}{3}a^{\dagger 3} + 4a^\dagger a^2). \quad (\text{A5})$$

The sixth term is

$$\frac{\xi^5}{5!}[A, [A, [A, [A, [A, B]]]]] = -\frac{(it\omega)^5}{5!}a + \frac{(it)^5}{5!}\hbar m\omega^6\beta(-5a + a^\dagger + \frac{121}{3}a^3 - \frac{61}{3}a^{\dagger 3} - 5a^\dagger a^2 + a^{\dagger 2}a). \quad (\text{A6})$$

It can be seen that, coefficients of each term are actually terms of expansion of a specific series and can be collected into a simple form. The terms without contribution of β is nothing but the usual results of ordinary quantum mechanics,

$$1 - i\omega + \frac{(i\omega)^2}{2!} - \frac{(i\omega)^3}{3!} + \frac{(i\omega)^4}{4!} - \frac{(i\omega)^5}{5!} + \dots = \sum_{n=0}^{\infty} \frac{(-i\omega)^n}{n!} = e^{-i\omega t}. \quad (\text{A7})$$

The terms proportional to β are induced by quantum gravitation. Surprisingly, collection of coefficients for each operators can be simplified either into exponential function or trigonometric function. Specifically, collection of coefficients of $a + a^\dagger a^2$ has the form

$$\begin{aligned} & \hbar m \omega \beta \left(-i\omega + \frac{(i\omega)^2}{2!} \times 2 - \frac{(i\omega)^3}{3!} \times 3 + \frac{(i\omega)^4}{4!} \times 4 - \frac{(i\omega)^5}{5!} \times 5 + \dots \right) \\ &= \hbar m \omega \beta \sum_{n=1}^{\infty} \frac{(-i\omega)^n}{(n-1)!} = -i\hbar m \omega^2 \beta e^{-i\omega t}. \end{aligned} \quad (\text{A8})$$

For $a^\dagger + a^{\dagger 2}a$, the result is

$$\hbar m \omega \beta \left(i\omega + \frac{(i\omega)^3}{3!} + \frac{(i\omega)^5}{5!} + \dots \right) = \hbar m \omega \beta \sum_{n=0}^{\infty} \frac{(i\omega)^{2n+1}}{(2n+1)!} = i\hbar m \omega \beta \sin \omega t. \quad (\text{A9})$$

Collection of coefficients of a^3 turns out to be

$$\begin{aligned} & \hbar m \omega \beta \left(i\omega \times \frac{1}{3} + \frac{(i\omega)^2}{2!} \times \left(-\frac{4}{3}\right) + \frac{(i\omega)^3}{3!} \times \frac{13}{3} + \frac{(i\omega)^4}{4!} \times \left(-\frac{40}{3}\right) + \frac{(i\omega)^5}{5!} \times \frac{121}{3} + \dots \right) \\ &= -\hbar m \omega \beta \sum_{n=1}^{\infty} \frac{(-i\omega)^n}{n!} \times \frac{3^n - 1}{3 \times 2} = \hbar m \omega \beta \frac{1}{6} (e^{-i\omega t} - e^{-3i\omega t}). \end{aligned} \quad (\text{A10})$$

Collection of coefficients of its complex conjugation term $a^{\dagger 3}$ is simplified to

$$\begin{aligned} & \hbar m \omega \beta \left(i\omega \times \left(-\frac{1}{3}\right) + \frac{(i\omega)^2}{2!} \times \left(-\frac{2}{3}\right) + \frac{(i\omega)^3}{3!} \times \left(-\frac{7}{3}\right) + \frac{(i\omega)^4}{4!} \times \left(-\frac{20}{3}\right) + \frac{(i\omega)^5}{5!} \times \left(-\frac{61}{3}\right) + \dots \right) \\ &= -\hbar m \omega \beta \sum_{n=1}^{\infty} \frac{(i\omega)^n}{n!} \frac{3^n - (-1)^n}{3 \times 4} = \hbar m \omega \beta \frac{1}{12} (e^{-i\omega t} - e^{3i\omega t}). \end{aligned} \quad (\text{A11})$$

To be more convincing, we have calculated the next six terms to verify the correctness of the collection of coefficients. It turns out that those twelve terms all conform with the series expansion very well. Thus, we get the time evolution of annihilation operator in Heisenberg picture, that is Eq. (7).

Appendix B: Expectation value of quadrature operators

According to Eq. (7), we have the following expectation values in a coherent state $|\alpha\rangle$

$$\begin{aligned} \langle X_1^2 \rangle = & \frac{1}{4}(1 + e^{-2it\omega}\alpha^2 + 2\alpha\alpha^* + e^{2it\omega}\alpha^{*2}) + \frac{1}{48}e^{-4it\omega}\hbar m\omega\beta[-6\alpha^4 - 6e^{8it\omega}\alpha^{*4} + \\ & e^{6it\omega}(-6 + 27\alpha^2 + 2\alpha^3\alpha^* + 9(1 + 4it\omega)\alpha^{*2} + 6(1 + 4it\omega)\alpha\alpha^{*3} + 4\alpha^{*4} - 48\alpha\text{Re}[\alpha] \\ & + e^{2it\omega}(-6 + \alpha^2(33 - 36it\omega - 2\alpha^2) - 24it\omega\alpha^3\alpha^* + 3\alpha^{*2} + 2\alpha\alpha^{*3} + 12\alpha(-4 + \alpha^2)\text{Re}[\alpha] \\ & + 2e^{4it\omega}(6 + 5\alpha^2(-6 + \alpha^2) + \alpha^{*2}(-6 - 4\alpha\alpha^* + \alpha^{*2} - 8\alpha(-6 + \alpha^2)\text{Re}[\alpha] \\ & + 24Abs[\alpha]^4\sin^2\omega t)], \end{aligned} \quad (\text{B1})$$

$$\begin{aligned} (\langle X_1 \rangle)^2 = & \frac{1}{4}e^{-2i\omega t}(\alpha + e^{2i\omega t}\alpha^*)^2 + \frac{1}{24}e^{-4i\omega t}\hbar m\omega\beta(\alpha + e^{2i\omega t}\alpha^*)[-3\alpha^3 - 3e^{6i\omega t}\alpha^{*3} \\ & + e^{2i\omega t}(6(1 - 2i\omega t)\alpha|\alpha|^2 - 6\alpha\alpha^{*2} + \alpha^{*3} + 2(\alpha(6 - 6i\omega t + \alpha^2) - 6\text{Re}[\alpha])) \\ & + e^{4i\omega t}(-12\alpha + \alpha^3 - 6\alpha|\alpha|^2 + 2\alpha^*(6i\omega t + \alpha^*(3\alpha + 6i\omega t\alpha + \alpha^*)) + 12\text{Re}[\alpha])], \end{aligned} \quad (\text{B2})$$

$$\begin{aligned} \langle X_2^2 \rangle = & \frac{1}{4}(1 - e^{-2it\omega}\alpha^2 + 2\alpha\alpha^* - e^{2it\omega}\alpha^{*2}) + \frac{1}{48}e^{-4it\omega}\hbar m\omega\beta[(2\alpha^4 - 3e^{6it\omega}(\alpha^2 - 2) \\ & + e^{2it\omega}(6 + 3\alpha^2(5 + 12i\omega t) - 4\alpha^4) + 2e^{4it\omega}(-6 - 6\alpha^2 + \alpha^4) + 12e^{2it\omega}(e^{2it\omega} - 1)^2|\alpha|^4 \\ & + 3e^{2it\omega}\alpha^{*2}(-1 - 4e^{2it\omega} + e^{4it\omega}(5 - 12i\omega t)) + 2e^{4it\omega}\alpha^{*4}(e^{2it\omega} - 1)^2 \\ & - 2e^{2it\omega}|\alpha|^2(-12 - 5\alpha^2 - 12i\omega t\alpha^2 + e^{4it\omega}(\alpha^2 - 12) + 4e^{2it\omega}(6 + \alpha^2) + (1 + 4e^{2it\omega} \\ & + e^{4it\omega}(12i\omega t - 5))\alpha^{*2})], \end{aligned} \quad (\text{B3})$$

$$\begin{aligned} (\langle X_2 \rangle)^2 = & -\frac{1}{4}e^{-2i\omega t}(\alpha + e^{2i\omega t}\alpha^*)^2 + \frac{1}{24}e^{-4i\omega t}\hbar m\omega\beta(\alpha - e^{2i\omega t}\alpha^*)[\alpha^3 + 2e^{2it\omega}\alpha(3 + 6i\omega t - \alpha^2) \\ & + e^{4it\omega}(\alpha^2 - 6) + 6e^{2it\omega}(1 + e^{2it\omega}\alpha^*(2i\omega t - 1)) - e^{2it\omega}\alpha^{*3}(e^{2it\omega} - 1)^2 \\ & - 6e^{2it\omega}|\alpha|^2((-1 + e^{2it\omega} - 2i\omega t)\alpha + (-1 + e^{2it\omega}(1 - 2i\omega t))\alpha^*)], \end{aligned} \quad (\text{B4})$$

$$\begin{aligned} \langle \hat{p}^2 \rangle = & 2\hbar m \langle X_2^2 \rangle + \frac{8}{3}\hbar^2\omega^2 m^2 \beta \langle X_2^4 \rangle \\ = & \frac{1}{2}\hbar m(1 - e^{-2i\omega t}\alpha^2 + 2|\alpha|^2 - e^{2i\omega t}\alpha^{*2}) + \\ & \frac{1}{24}e^{-4i\omega t}\hbar^2\omega^2 m^2 \beta[(6e^{2i\omega t} + 6e^{6i\omega t} - 3e^{2i\omega t}(3 + 4e^{2i\omega t} - 12it\omega)\alpha^2 + 2(3 - 2e^{2i\omega t} + e^{4i\omega t})\alpha^4 \\ & + 12e^{2i\omega t}(1 + e^{4i\omega t})|\alpha|^4 + 16e^{2i\omega t}|\alpha|^2(3e^{2i\omega t} - \alpha^2) + (-3e^{2i\omega t} - 12e^{4i\omega t} - e^{6i\omega t}(9 + 36it\omega))\alpha^{*2} \\ & - 16e^{6i\omega t}|\alpha|^2\alpha^{*2} + 2e^{4i\omega t}(1 - 2e^{2i\omega t} + 3e^{4i\omega t})\alpha^{*4} - 2e^{2i\omega t}|\alpha|^2(-12(e^{2i\omega t} - 1)^2 + (-5 + 4e^{2i\omega t} \\ & + e^{4i\omega t} - 12it\omega)\alpha^2 + (1 + 4e^{2i\omega t} + e^{4i\omega t}(-5 + 12it\omega)\alpha^{*2})]. \end{aligned} \quad (\text{B5})$$

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